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Continuous-time additive Hopfield-type neural networks with impulses

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Abstract

We investigate the global stability characteristics of a system of equations modelling the dynamics of additive Hopfield-type neural networks with impulses in the continuous-time case.

Keywords: Impulsive equations; Additive Hopfield-type neural networks; Global stability

1. Introduction

Anyone can see that the human brain is superior to a digital computer at many tasks. For example, from the processing of visual information point of view; a one-year-old baby is much better and faster at recognizing objects, faces, and so on than even the most advanced fastest supercomputer systems. The following reasons are the real motivation for studying neural computation [8]. It is an alternative computational paradigm to the usual one (based on a programmed instruction sequence), which was introduced by von Neumann [17] and

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has been used as the basis of almost all machine computation to date. The brain has many other features that would be desirable in artificial systems:

- It is robust and fault tolerant. Nerve cells in the brain die every day without affecting its performance significantly.
- It is flexible. It can easily adjust to a new environment by "learning," no need to be programmed in Pascal, Fortran or C+ and so on.
- It can deal with information that is fuzzy, probabilistic, noisy, or inconsistent.
- It is highly parallel.
- It is small, compact, and dissipates very little power.

Artificial neural networks are computational paradigms which implement simplified models of their biological counterparts, biological neural networks. Biological neural networks are the local assemblages of neurons and their dendritic connections that form the (human) brain. Accordingly, artificial neural networks are characterized by

- local processing in artificial neurons (or processing elements),
- massively parallel processing, implemented by rich connection pattern between processing elements,
- the ability to acquire knowledge via learning from experience,
- knowledge storage in distributed memory, the synaptic processing element connections.

The attempt of implementing neural networks for brain-like computations like patterns recognition, decisions making, motory control and many others is made possible by the advent of large scale computers in the late 1950's. Indeed, artificial neural networks can be viewed as a major new approach to computational methodology since the introduction of digital computers.

Although the initial intent of artificial neural networks was to explore and reproduce human information processing tasks such as speech, vision, and knowledge processing, artificial neural networks also demonstrated their superior capability for classification and function approximation problems. This has great potential for solving complex problems such as systems control, data compression, optimization problems, pattern recognition, and system identification.

Hopfield-type (additive) networks have been studied intensively during the last two decades and have been applied to optimization problems [5–8,14]. The original model used two-state threshold "neurons" that followed a stochastic algorithm: each model neuron i had two states, characterized by the values V_i^0 or V_i^1 (which may often be taken as 0 and 1, respectively). The input of each neuron came from two sources, external inputs I_i and inputs from other neurons. The total input to neuron i is then

$$\text{Input to } i = H_i = \sum_{i \neq j} T_{ij} V_j + I_i,$$

where T_{ij} can be biologically viewed as a description of the synaptic interconnection strength from neuron j to neuron i. The motion of the state of a system of N neurons in state space describes the computation that the set of neurons is performing. A model therefore must describe how the state evolves in time, and the original model describes this in terms of a stochastic evolution. Each neuron samples its input at random times. It changes the value of its output or leaves it fixed according to a threshold rule with thresholds U_i [9,10],

$$V_i \to V_i^0 \quad \text{if } \sum_{i\neq j} T_{ij} V_j + I_i < U_i,$$

$$V_i \to V_i^1 \quad \text{if } \sum_{i\neq j} T_{ij} V_j + I_i > U_i.$$

In order to solve problems in the fields of optimization, neural control and signal processing, neural networks have to be designed such that there is only one equilibrium point and this equilibrium point is globally asymptotically stable so as to avoid the risk of having spurious equilibria and local minima. In the case of global stability, there is no need to be specific about the initial conditions for the neural circuits since all trajectories starting from anywhere settle down at the same unique equilibrium. If the equilibrium is exponentially asymptotically stable, the convergence is fast for real-time computations. The unique equilibrium depends on the external stimulus. The nonlinear neural activation functions $f_i(\cdot)$, $i \in \mathbb{Z}^+$, are usually chosen to be continuous and differentiable nonlinear sigmoid functions satisfying the following conditions:

(a) $f_i(x) \to \mp 1$ as $x \to \mp\infty$;
(b) $f_i(x)$ is bounded above by 1 and below by -1;
(c) $f_i(x) = 0$ at a unique point $x = 0$;
(d) $f_i'(x) > 0$ and $f_i'(x) \to 0$ as $x \to \mp\infty$;
(e) $f_i'(x)$ has a global maximum value of 1 at the unique point $x = 0$.

Some examples of activation functions $f_i(\cdot)$ are

$$f_i(x) = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad f_i(x) = \frac{1 - e^{-x}}{1 + e^{-x}} = \tanh\left(\frac{x}{2}\right),$$

$$f_i(x) = \frac{2}{\pi}\arctan\left(\frac{\pi}{2}x\right), \qquad f_i(x) = \frac{x^2}{1 + x^2}\operatorname{sgn}(x),$$

where $\operatorname{sgn}(\cdot)$ is a signum function and all the above nonlinear functions are bounded, monotonic and nondecreasing functions. It has been shown that the absolute capacity of an associative memory network can be improved by replacing the usual sigmoid activation functions. There, it seems appropriate that nonmonotonic functions might be better candidates for neuron activation in designing and implementing an artificial neural network. In many electronic circuits, amplifiers that have neither monotonically increasing nor continuously differentiable input–output functions are frequently adapted.

Henceforth we assume that each activation function $f_i(\cdot)$, $i = 1, 2, \ldots, m$, satisfies the following conditions:

(H1) $f_i : \mathbb{R} \to \mathbb{R}$ is Lipschitzian with Lipschitz constant $L_i > 0$,

$$\left| f_i(x) - f_i(y) \right| \leqslant L_i |x - y| \quad \text{for all } x, y \in \mathbb{R}.$$

(H2) $|f_i(x)| \leqslant M_i$, $x \in \mathbb{R}$, for some constant $M_i > 0$.

This type of activation functions is clearly more general than the usual sigmoid activation functions.

In [13] the global stability characteristic of a system of equations modelling the dynamics of additive Hopfield-type neural networks both in the continuous and discrete-time cases is investigated. In particular, a novel method of obtaining a discrete-time dynamical system whose dynamics is inherited from the continuous-time dynamical system is studied. This aspect is important since numerical algorithms of Hopfield-type differential equations lead to discrete-time dynamic systems and such discrete-time systems should not give rise to any spurious behaviour if either system is to be used for coding equilibrium as associative memories corresponding to temporally uniform external stimuli obtained. The discrete-time models serve as global numerical methods on unbounded intervals for the continuous-time systems [12].

In the present paper we investigate the global stability characteristics of these systems supplemented with impulse conditions in the continuous-time case. The presence of impulses requires some modifications and the imposing of additional conditions on the systems. The investigation of the stability of the respective discrete systems with impulse effect is an object of another paper [1].

2. Main results

Consider the following Hopfield-type model of neural network with impulses:

$$\begin{aligned} &\frac{dx_i}{dt} = -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j\big(x_j(t)\big) + c_i, \quad t > 0,\ t \neq t_k, \\ &\Delta x_i(t_k) = I_i\big(x_i(t_k)\big), \quad i \in \{1, \ldots, m\},\ k = 1, 2, \ldots, \end{aligned} \tag{2.1}$$

where $\Delta x(t_k) = x(t_k + 0) - x(t_k - 0)$ are the impulses at moments t_k and $t_1 < t_2 < \cdots$ is a strictly increasing sequence such that $\lim_{k\to\infty} t_k = +\infty$; $x_i(t)$ corresponds to the membrane potential of the unit i at time t; $f_j(\cdot)$ denotes a measure of response or activation to its incoming potentials; b_{ij} denotes the synaptic connection weight of the unit j on the unit i; the constants c_i correspond to the external bias or input from outside the network to the unit i; the coefficient a_i is the rate with which the unit self-regulates or resets its potential when isolated from other units and inputs. We refer for more detail to [13,15] and references cited therein.

As usual in the theory of impulsive differential equations, at the points of discontinuity t_k of the solution $t \mapsto x_i(t)$ we assume that $x_i(t_k) \equiv x_i(t_k - 0)$. It is clear that, in general, the derivatives $\dot{x}_i(t_k)$ do not exist. On the other hand, according to the first equality of (2.1) there exist the limits $\dot{x}_i(t_k \mp 0)$. According to the above convention, we assume $\dot{x}_i(t_k) \equiv \dot{x}_i(t_k - 0)$.

First, we will establish sufficient conditions for all solutions of the impulsive autonomous differential system to converge at an exponential rate to an equilibrium.

Theorem 2.1. *Assume that conditions* (H1) *and* (H2) *are satisfied. Suppose further that*

(i) $a_i > 0$, $b_{ij} \in \mathbb{R}$, $c_i \in \mathbb{R}$, $i, j \in \{1, \dots, m\}$,
(ii) *the following inequalities hold*:

$$a_i - L_i \sum_{j=1}^{m} |b_{ji}| > 0, \quad i \in \{1, \dots, m\},$$

(iii) *the impulsive operators* $I_i(x_i(t))$ *satisfy*

$$I_i\big(x_i(t_k)\big) = -\gamma_{ik}\big(x_i(t_k) - x_i^*\big), \quad 0 < \gamma_{ik} < 2,\ i \in \{1, \dots, m\},\ k \in \mathbb{Z}^+.$$

Then the equilibrium $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_m^*)^T$ *of* (2.1) *is unique.*

The proof of the theorem is similar to that given in [13, Theorem 2.1] or [15, Theorem 1.2.1]. An additional difference is the consideration of the impulse effect. The equilibrium obviously satisfies

$$a_i x_i^* = \sum_{j=1}^{m} b_{ij} f_j\big(x_j^*\big) + c_i, \quad i \in \{1, \dots, m\}. \tag{2.2}$$

Several authors have obtained sufficient conditions for global stability and exponential stability of equilibria of continuous-time Hopfield-type networks.

We are interested in obtaining sufficient conditions for the global exponential asymptotic stability of equilibria in continuous-time Hopfield-type networks in the presence of impulses.

Theorem 2.2. *Assume that all conditions of Theorem* 2.1 *hold. Then there exists a constant* $\alpha > 0$ *such that all solutions of* (2.1) *satisfy the inequality*

$$\sum_{i=1}^{m} \big|x_i(t) - x_i^*\big| \leqslant e^{-\alpha t} \sum_{i=1}^{m} \big|x_i(0) - x_i^*\big|, \quad t > 0,$$

where $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_m^*)^T$ *denotes the equilibrium.*

Proof. The proof of the theorem follows like that of [13, Theorem 3.1] or [15, Theorem 1.2.2]. We have from (2.1) that

$$\frac{d}{dt}\big(x_i(t) - x_i^*\big) = -a_i\big(x_i(t) - x_i^*\big) + \sum_{j=1}^{m} b_{ij}\big(f_j\big(x_j(t)\big) - f_j\big(x_j^*\big)\big)$$

for $i \in \{1, \dots, m\}$, $t > 0$, $t \neq t_k$, $k \in \mathbb{Z}^+$, and hence by condition (H1),

$$\frac{d^+}{dt}\big|x_i(t) - x_i^*\big| \leqslant -a_i\big|x_i(t) - x_i^*\big| + \sum_{j=1}^{m} |b_{ij}| L_j \big|x_j(t) - x_j^*\big| \tag{2.3}$$

for $i \in \{1, \dots, m\}$, $t > 0$, $t \neq t_k$, $k \in \mathbb{Z}^+$, where d^+/dt denotes the upper right derivative. Also,

$$x_i(t_k + 0) - x_i^* = x_i(t_k) + I_i\big(x(t_k)\big) - x_i^* = (1 - \gamma_{ik})\big(x_i(t_k) - x_i^*\big),$$

thus

$$\left|x_i(t_k + 0) - x_i^*\right| = |1 - \gamma_{ik}|\left|x_i(t_k) - x_i^*\right| \leqslant \left|x_i(t_k) - x_i^*\right|$$

for $i \in \{1, \dots, m\}$, $k \in \mathbb{Z}^+$.

We define a Lyapunov function $V(\cdot)$ by

$$V(t) = V(x_1, x_2, \dots, x_m)(t) = \sum_{i=1}^{m}\left|x_i(t) - x_i^*\right|$$

for $t \geqslant 0$ and by virtue of (2.3) we can obtain

$$\begin{aligned}
\frac{d^+V(t)}{dt} &= \sum_{i=1}^{m} \frac{d^+}{dt}\left|x_i(t) - x_i^*\right| \\
&\leqslant \sum_{i=1}^{m}\left(-a_i\left|x_i(t) - x_i^*\right| + \sum_{j=1}^{m}|b_{ij}|L_j\left|x_j(t) - x_j^*\right|\right) \\
&= -\sum_{i=1}^{m}\left(a_i - L_i\sum_{j=1}^{m}|b_{ji}|\right)\left|x_i(t) - x_i^*\right|.
\end{aligned}$$

By virtue of condition (ii) of Theorem 2.1 there exists a real number $\alpha > 0$ such that

$$a_i - L_i\sum_{j=1}^{m}|b_{ji}| \geqslant \alpha, \quad i \in \{1, \dots, m\},$$

and it follows that

$$\frac{d^+V(t)}{dt} \leqslant -\alpha V(t), \quad t > 0,\ t \neq t_k. \tag{2.4}$$

Also,

$$V(t_k + 0) = \sum_{i=1}^{m}\left|x_i(t_k + 0) - x_i^*\right| \leqslant \sum_{i=1}^{m}\left|x_i(t_k) - x_i^*\right| = V(t_k), \quad k \in \mathbb{Z}^+. \tag{2.5}$$

Then using the exponential stability theorem [4,11,16] and (2.4), (2.5), we get

$$\frac{d^+V}{dt} \leqslant -\alpha V(t) \quad \Rightarrow \quad V(t) \leqslant e^{-\alpha t}V(0) \quad \forall t > 0.$$

So,

$$\sum_{i=1}^{m}\left|x_i(t) - x_i^*\right| \leqslant e^{-\alpha t}\sum_{i=1}^{m}\left|x_i(0) - x_i^*\right| \quad \forall t > 0$$

and this completes the proof of the theorem. $\square$

The next theorem represents a generalization of Theorem 2.2.

Theorem 2.3. *Assume that conditions* (H1) *and* (H2) *are satisfied. Suppose further that*

(i) $a_i > 0$, $b_{ij} \in \mathbb{R}$, $c_i \in \mathbb{R}$, $i, j \in \{1, \dots, m\}$,
(ii) *the number* $i(0,t)$ *of moments of impulse effect between* 0 *and* t *satisfies*

$$\limsup_{t\to+\infty} \frac{i(0,t)}{t} = p < +\infty,$$

(iii) *the impulsive operators* $I_i(x_i(t))$ *satisfy*

$$\left|I_i\big(x_i(t_k)\big)\right| \leqslant c\left|x_i(t_k) - x_i^*\right|, \quad i \in \{1, \dots, m\},\ k \in \mathbb{Z}^+,$$

for some positive constant c,
(iv) *there exists a real number* $\alpha > 0$ *such that*

$$a_i - L_i \sum_{j=1}^{m} |b_{ji}| - p\ln(1+c) \geqslant \alpha, \quad i \in \{1, \dots, m\}.$$

Then for any number $\tilde{\alpha}$ *satisfying* $0 < \tilde{\alpha} < \alpha$ *and any compact neighbourhood* K *of the equilibrium* $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_m^*)^T$ *there exists a constant* $\beta \geqslant 1$ *such that all solutions* $\mathbf{x}(t)$ *of* (2.1) *with* $\mathbf{x}(0) \in K$ *satisfy the inequality*

$$\sum_{i=1}^{m}\left|x_i(t) - x_i^*\right| \leqslant \beta e^{-\tilde{\alpha}t} \sum_{i=1}^{m}\left|x_i(0) - x_i^*\right|, \quad t > 0. \tag{2.6}$$

Proof. We need to just slightly modify the proof of Theorem 2.2. Now inequalities (2.4) and (2.5) should be replaced by

$$\frac{d^+V}{dt} \leqslant -\big(\alpha + p\ln(1+c)\big)V(t), \quad t > 0,\ t \neq t_k,$$
$$V(t_k + 0) \leqslant (1+c)V(t_k), \quad k \in \mathbb{Z}^+,$$

which implies that

$$V(t) \leqslant (1+c)^{i(0,t)} e^{-(\alpha + p\ln(1+c))t} V(0)$$

or

$$V(t) \leqslant (1+c)^{i(0,t)-pt} e^{-\alpha t} V(0).$$

For any $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that the inequality

$$\frac{i(0,t)}{t} \leqslant p + \varepsilon$$

is satisfied for all $t \geqslant T$. For such t we have $i(0,t) \leqslant (p+\varepsilon)t$ and

$$V(t) \leqslant (1+c)^{\varepsilon t} e^{-\alpha t} V(0)$$

or

$$V(t) \leqslant e^{-(\alpha - \varepsilon\ln(1+c))} V(0).$$

Let $\tilde{\alpha}$ be an arbitrary number satisfying $0 < \tilde{\alpha} < \alpha$. If we take $\alpha - \varepsilon \ln(1 + c) = \tilde{\alpha}$, i.e., $\varepsilon = (\alpha - \tilde{\alpha})/\ln(1 + c)$, then

$$V(t) \leqslant e^{-\tilde{\alpha}t} V(0) \quad \forall t \geqslant T.$$

There exists a constant $\beta \geqslant 1$ such that

$$V(t) \leqslant \beta e^{-\tilde{\alpha}t} V(0) \quad \forall t > 0.$$

The last inequality implies (2.6) in which β may depend on the solution $\mathbf{x}(t)$, i.e., on the initial condition $\mathbf{x}(0)$. Because of the compactness of K we can choose β so that (2.6) is valid for all $\mathbf{x}(0) \in K$. $\square$

It is implicitly assumed in the formulation of system (2.1) that the neurons process input and produce output instantaneously. It is known, however, that such instantaneous processing and delivery is not always true and there are significant time delays both in neural processing and axonal transmission. So we will generalize system (2.1) by inserting time delays in neural networks. Then, we will consider the system

$$\frac{dx_i}{dt} = -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j\big(x_j(t - \tau_{ij})\big) + c_i, \quad t > 0,\ t \neq t_k, \tag{2.7}$$

in which $i \in \{1, 2, \ldots, m\}$ and $\tau_{ij} \geqslant 0$ corresponds to the transmission delay for $i, j \in \{1, 2, \ldots, m\}$. The impulsive conditions are

$$\Delta x_i(t) = I_i\big(x_i(t)\big), \quad t = t_k,\ k = 1, 2, \ldots. \tag{2.8}$$

This system is supplemented with initial functions of the form

$$x_i(s) = \psi_i(s), \quad s \in [-\tau, 0],\ i \in \{1, \ldots, m\},\ \tau = \max_{i,j \in \{1,\ldots,m\}} \{\tau_{ij}\},$$

where $\psi_i(s)$ is continuous for $s \in [-\tau, 0]$.

Theorem 2.4. *Suppose that the conditions of Theorem* 2.1 *hold. Let* $\tau_{ij} \geqslant 0$ *for* $i, j \in \{1, \ldots, m\}$. *Then there exist constants* $\beta \geqslant 1$ *and* $\varepsilon > 0$ *such that all solutions of* (2.7), (2.8) *satisfy*

$$\sum_{i=1}^{m} \big|x_i(t) - x_i^*\big| \leqslant \beta e^{-\varepsilon t} \sum_{i=1}^{m} \Big(\sup_{s \in [-\tau, 0]} \big|x_i(s) - x_i^*\big|\Big), \quad t > 0.$$

Proof. The equilibrium $\mathbf{x}^*$ of (2.7), (2.8) is unique because of the condition (H1) and Theorem 2.1. Again its components satisfy Eq. (2.2). Let $F_i(\cdot)$, $i \in \{1, \ldots, m\}$, be defined by

$$F_i(\varepsilon_i) = a_i - \varepsilon_i - L_i \sum_{j=1}^{m} |b_{ji}| e^{\varepsilon_i \tau_{ji}},$$

where $\varepsilon_i \in [0, \infty)$, $i \in \{1, \ldots, m\}$. It is obvious that

$$F_i(0) = a_i - L_i \sum_{j=1}^{m} |b_{ji}| > 0$$

for all $i \in \{1, \dots, m\}$. Since $F_i(\cdot)$ is continuous on $[0, \infty)$ and $F_i(\varepsilon_i) \to -\infty$ as $\varepsilon_i \to \infty$, there exists $\varepsilon_i^* > 0$ such that $F_i(\varepsilon_i^*) = 0$ and $F_i(\varepsilon_i) > 0$ for $\varepsilon_i \in (0, \varepsilon_i^*)$. By choosing $\varepsilon = \min\{\varepsilon_1^*, \varepsilon_2^*, \dots, \varepsilon_m^*\}$, we obtain

$$F_i(\varepsilon) = a_i - \varepsilon - L_i \sum_{j=1}^{m} |b_{ji}| e^{\varepsilon \tau_{ji}} \geqslant 0$$

for all $i \in \{1, \dots, m\}$. Therefore from (2.7) we have

$$\frac{d^+}{dt}\left|x_i(t) - x_i^*\right| \leqslant -a_i\left|x_i(t) - x_i^*\right| + \sum_{j=1}^{m} |b_{ij}| L_j \left|x_j(t - \tau_{ij}) - x_j^*\right| \tag{2.9}$$

for $i \in \{1, \dots, m\}$ and $t > 0$, $t \neq t_k$. Now let us define

$$y_i(t) = e^{\varepsilon t}\left|x_i(t) - x_i^*\right|, \quad i \in \{1, \dots, m\},\ t \in [-\tau, \infty), \tag{2.10}$$

and from (2.9), (2.10) we derive

$$\begin{aligned} &\frac{d^+ y_i(t)}{dt} \leqslant -(a_i - \varepsilon) y_i(t) + \sum_{j=1}^{m} |b_{ij}| L_j e^{\varepsilon \tau_{ij}} y_j(t - \tau_{ij}), \\ &\quad t > 0,\ t \neq t_k,\ k \in \mathbb{Z}^+, \end{aligned} \tag{2.11}$$

for $i \in \{1, \dots, m\}$. Also,

$$y_i(t_k + 0) = |1 - \gamma_{ik}| y_i(t_k) \leqslant y_i(t_k).$$

We consider a Lyapunov functional $V(\cdot)$ defined by

$$V(t) = \sum_{i=1}^{m}\left(y_i(t) + \sum_{j=1}^{m} |b_{ij}| L_j e^{\varepsilon \tau_{ij}} \int_{t-\tau_{ij}}^{t} y_j(s)\, ds\right), \tag{2.12}$$

and we note that $V(t) > 0$ for $t > 0$ and $V(0)$ is positive and finite. Calculating the rate of change of $V(t)$ along the solutions of (2.7), we obtain

$$\begin{aligned} \frac{d^+ V(t)}{dt} &\leqslant \sum_{i=1}^{m}\left(-(a_i - \varepsilon) y_i(t) + \sum_{j=1}^{m} |b_{ij}| L_j e^{\varepsilon \tau_{ij}} y_j(t)\right) \\ &= -\sum_{i=1}^{m}\left(a_i - \varepsilon - L_i \sum_{j=1}^{m} |b_{ji}| e^{\varepsilon \tau_{ji}}\right) y_i(t) \leqslant 0 \end{aligned}$$

for $t > 0$, $t \neq t_k$. Also, as in the proof of Theorem 2.2 we obtain that $V(t_k + 0) \leqslant V(t_k)$ for $k \in \mathbb{Z}^+$. It follows that $V(t) \leqslant V(0)$ for $t > 0$ and hence from (2.11) and (2.12) we obtain

$$\begin{aligned} \sum_{i=1}^{m} y_i(t) &\leqslant \sum_{i=1}^{m}\left(y_i(0) + \sum_{j=1}^{m} |b_{ij}| L_j e^{\varepsilon \tau_{ij}} \int_{-\tau_{ij}}^{0} y_j(s)\, ds\right) \\ &= \sum_{i=1}^{m}\left(y_i(0) + L_i \sum_{j=1}^{m} |b_{ji}| e^{\varepsilon \tau_{ji}} \int_{-\tau_{ji}}^{0} y_i(s)\, ds\right), \end{aligned} \tag{2.13}$$

where $t > 0$. Then using (2.10) and from (2.13) it follows that

$$\sum_{i=1}^{m}\left|x_i(t) - x_i^*\right| \leqslant e^{-\varepsilon t}\sum_{i=1}^{m}\left(1 + L_i\sum_{j=1}^{m}|b_{ji}|\tau_{ji}e^{\varepsilon\tau_{ji}}\right)\sup_{s\in[-\tau,0]}\left|x_i(s) - x_i^*\right|$$
$$\leqslant \beta e^{-\varepsilon t}\sum_{i=1}^{m}\left(\sup_{s\in[-\tau,0]}\left|x_i(s) - x_i^*\right|\right),$$

where

$$t > 0 \quad \text{and} \quad \beta = \max_{i\in\{1,\dots,m\}}\left(1 + L_i\sum_{j=1}^{m}|b_{ji}|\tau_{ji}e^{\varepsilon\tau_{ji}}\right) \geqslant 1.$$

Here $\mathbf{x}(t) = (x_1(t), x_2(t), \dots, x_m(t))^T$ denotes an arbitrary solution of (2.7), (2.8) and we can conclude that the equilibrium $\mathbf{x}^*$ is globally exponentially stable and this completes the proof of the theorem. □

In the model (2.7) we have assumed that the time delays are discrete. While this assumption is not unreasonable, a more satisfactory hypothesis is that the time delays are continuously distributed over a certain duration of time. We shall modify the system (2.7) to a system of integro-differential equations [2,3,13] of the form

$$\frac{dx_i(t)}{dt} = -a_ix_i(t) + \sum_{j=1}^{m}b_{ij}f_j\left(\int_{-\infty}^{t}K_{ij}(t-s)x_j(s)\,ds\right) + c_i,$$
$$i = 1, 2, \dots, m,\ t > 0,$$

or

$$\frac{dx_i(t)}{dt} = -a_ix_i(t) + \sum_{j=1}^{m}b_{ij}f_j\left(\int_{0}^{\infty}K_{ij}(s)x_j(t-s)\,ds\right) + c_i,$$
$$i = 1, 2, \dots, m,\ t > 0, \tag{2.14}$$

where for $i, j \in \{1, \dots, m\}$ the delay kernels $K_{ij}(s)$ are assumed to satisfy the following conditions:

(H3) $K_{ij} : [0, \infty) \to [0, \infty)$ are bounded and continuous.

(H4) $\int_0^\infty K_{ij}(s)\,ds = 1$.

(H5) There exists a positive number μ such that

$$\int_{0}^{\infty}K_{ij}(s)e^{\mu s}\,ds < \infty.$$

For an integro-differential equation an impulsive condition including both the functional value and its integral also seems natural. We take the impulse conditions in the form

$$\Delta x_i(t_k) = I_i\big(x_i(t_k)\big) = B_{ik}x_i(t_k) + \int\limits_{t_{k-1}}^{t_k} c_{ik}(s)x_i(s)\,ds + \alpha_{ik}, \quad k \in \mathbb{Z}^+, \tag{2.15}$$

where $t_k > t_0 = 0$ and $c_{ik} : [t_{k-1}, t_k] \to \mathbb{R}$ are measurable functions, essentially bounded on the respective interval, B_{ik} and α_{ik} are some real constants. For more details about impulse conditions of this and more general form see [2] and references therein.

The initial conditions associated with system (2.14), (2.15) are given by

$$x_i(s) = \psi_i(s), \quad s \in (-\infty, 0],\ i = 1, 2, \ldots, m,$$

where $\psi_i(s)$ are bounded and continuous on $(-\infty, 0]$.

Theorem 2.5. *Suppose the conditions* (H1) *and* (H2) *and conditions* (i) *and* (ii) *of Theorem* 2.1 *hold. Additionally $K_{ij}(\cdot)$ satisfy the conditions* (H3)–(H5). *Then all solutions of system* (2.14), (2.15) *satisfy the estimate*

$$\sum_{i=1}^{m}\big|x_i(t) - x_i^*\big| \leqslant \beta e^{-\varepsilon t} \prod_{\nu=1}^{i(0,t)} \big\{1 + B_\nu\big(1 + \varphi(\varepsilon, t_\nu - t_{\nu-1})\big)\big\} \times \sum_{i=1}^{m} \sup_{w\in(-\infty,0]} \big|x_i(w) - x_i^*\big|$$

for all $t > 0$ and some $\varepsilon > 0$, where $\mathbf{x}^* = (x_1^*, \ldots, x_m^*)^T$ is an equilibrium,

$$\beta = \max_{1\leqslant i\leqslant m}\left(1 + L_i \sum_{j=1}^{m} |b_{ji}| \int\limits_0^\infty K_{ji}(s)se^{\varepsilon s}\,ds\right) \geqslant 1,$$

$$B_k = \max_{1\leqslant i\leqslant m} \max\big\{|B_{ik}|, \|c_{ik}\|_{L_\infty(t_{k-1},t_k)}\big\}, \tag{2.16}$$

and the function $\varphi(\varepsilon, z)$ is defined for $\varepsilon, z > 0$ by

$$\varphi(\varepsilon, z) = \frac{e^{\varepsilon z} - 1}{\varepsilon}. \tag{2.17}$$

Proof. First we notice that the components x_i^*, $i = 1, \ldots, m$, of the equilibrium point $\mathbf{x}^*$ satisfy Eq. (2.2) as well as

$$\left(B_{ik} + \int\limits_{t_{k-1}}^{t_k} c_{ik}(s)\,ds\right)x_i^* + \alpha_{ik} = 0, \quad k \in \mathbb{Z}^+. \tag{2.18}$$

It is clear that there can exist at most one equilibrium point. We suppose that it does exist.

We have from (2.14) that

$$\frac{d^+}{dt}\left|x_i(t)-x_i^*\right| \leqslant -a_i\left|x_i(t)-x_i^*\right| + \sum_{j=1}^{m}|b_{ij}|L_j\int_0^\infty K_{ij}(s)\left|x_j(t-s)-x_j^*\right|ds, \tag{2.19}$$

where $i\in\{1,\dots,m\}$, $t>0$, $t\neq t_k$.

Let us consider the functions $F_i:[0,\mu]\to(-\infty,\infty)$ defined by

$$F_i(\varepsilon_i)=a_i-\varepsilon_i-L_i\sum_{j=1}^{m}|b_{ji}|\int_0^\infty K_{ji}(s)e^{\varepsilon_i s}\,ds,\quad \varepsilon_i\in[0,\mu],\ i=1,2,\dots,m.$$

Now, because of the assumptions (H3)–(H5) each $F_i(\cdot)$ is well defined on $[0,\mu]$. We have

$$F_i(0)=a_i-L_i\sum_{j=1}^{m}|b_{ji}|>0$$

by virtue of condition (ii) of Theorem 2.1. Since $F_i(\cdot)$ is continuous and decreasing on $[0,\mu]$, it follows that there exists $\varepsilon_i^*\in(0,\mu]$ such that $F_i(\varepsilon_i)>0$ for $\varepsilon_i\in(0,\varepsilon_i^*)$, $i\in\{1,\dots,m\}$. Choosing $\varepsilon^*=\min\{\varepsilon_1^*,\varepsilon_2^*,\dots,\varepsilon_m^*\}$, we have

$$\begin{aligned}&F_i(\varepsilon)=a_i-\varepsilon-L_i\sum_{j=1}^{m}|b_{ji}|\int_0^\infty K_{ji}(s)e^{\varepsilon s}\,ds>0,\\ &\quad\varepsilon\in(0,\varepsilon^*),\ i\in\{1,\dots,m\}.\end{aligned} \tag{2.20}$$

Correspondingly, define

$$y_i(t)=\left|x_i(t)-x_i^*\right|e^{\varepsilon t},$$

where $i\in\{1,\dots,m\}$, $t\in\mathbb{R}$, and from (2.19) we derive

$$\frac{d^+y}{dt}\leqslant-(a_i-\varepsilon)y_i(t)+\sum_{j=1}^{m}|b_{ij}|L_j\int_0^\infty K_{ij}(s)e^{\varepsilon s}y_j(t-s)\,ds,\quad t>0,\ t\neq t_k.$$

We define a Lyapunov functional $V(\cdot)$ by

$$V(t)=\sum_{i=1}^{m}\left\{y_i(t)+\sum_{j=1}^{m}|b_{ij}|L_j\int_0^\infty K_{ij}(s)e^{\varepsilon s}\left(\int_{t-s}^{t}y_j(w)\,dw\right)ds\right\},\quad t\geqslant 0.$$

It is found that $V(t)\geqslant 0$ for $t>0$ and that

$$V(0)\leqslant\sum_{i=1}^{m}\left(y_i(0)+\sum_{j=1}^{m}|b_{ij}|L_j\int_0^\infty K_{ij}(s)e^{\varepsilon s}s\,ds\sup_{w\in(-\infty,0]}y_j(w)\right).$$

This implies that $0<V(0)<\infty$ since $\int_0^\infty K_{ij}(s)e^{\varepsilon s}s\,ds<\infty$ for $\varepsilon<\mu$.

We can now calculate

$$\frac{d^+V(t)}{dt} \leqslant \sum_{i=1}^{m}\left\{-(a_i-\varepsilon)y_i(t)+\sum_{j=1}^{m}|b_{ij}|L_j\left(\int_0^\infty K_{ij}(s)e^{\varepsilon s}\,ds\right)y_j(t)\right\}$$

$$= -\sum_{i=1}^{m}\left\{a_i-\varepsilon-L_i\sum_{j=1}^{m}|b_{ji}|\int_0^\infty K_{ji}(s)e^{\varepsilon s}\,ds\right\}y_i(t), \quad t>0,\ t\neq t_k.$$

As an application of (2.20) we have $d^+V/dt \leqslant 0$ for $t>0$, $t\neq t_k$. It follows that

$$V(t)\leqslant V(t_{k-1}+0) \quad \text{for } t\in(t_{k-1},t_k],\ k\in\mathbb{Z}^+. \tag{2.21}$$

Further on, making use of the equalities (2.18), for an arbitrary moment of impulse effect t_k, $k\in\mathbb{Z}^+$, we successively find

$$\Delta x_i(t_k) = B_{ik}\big(x_i(t_k)-x_i^*\big)+\int_{t_{k-1}}^{t_k} c_{ik}(s)\big(x_i(s)-x_i^*\big)\,ds,$$

$$\big|\Delta x_i(t_k)\big| \leqslant B_k\left(\big|x_i(t_k)-x_i^*\big|+\int_{t_{k-1}}^{t_k}\big|x_i(s)-x_i^*\big|\,ds\right),$$

$$\big|x_i(t_k+0)-x_i^*\big| \leqslant (1+B_k)\big|x_i(t_k)-x_i^*\big|+B_k\int_{t_{k-1}}^{t_k}\big|x_i(s)-x_i^*\big|\,ds,$$

$$y_i(t_k+0)\leqslant(1+B_k)y_i(t_k)+B_k\int_{t_{k-1}}^{t_k}e^{\varepsilon(t_k-s)}y_i(s)\,ds, \quad i=1,\dots,m,$$

and, making use of (2.21), we obtain with $\varphi(\varepsilon,z)$ given by (2.17),

$$V(t_k+0)\leqslant(1+B_k)V(t_k)+B_k\int_{t_{k-1}}^{t_k}e^{\varepsilon(t_k-s)}V(s)\,ds$$

$$\leqslant\left\{1+B_k\left(1+\int_{t_{k-1}}^{t_k}e^{\varepsilon(t_k-s)}\,ds\right)\right\}V(t_{k-1}+0)$$

$$=\left\{1+B_k\left[1+\frac{e^{\varepsilon(t_k-t_{k-1})}-1}{\varepsilon}\right]\right\}V(t_{k-1}+0)$$

$$\equiv\big\{1+B_k\big(1+\varphi(\varepsilon,t_k-t_{k-1})\big)\big\}V(t_{k-1}+0), \quad k\in\mathbb{Z}^+.$$

Applying successively the last inequality and (2.21), we derive

$$V(t)\leqslant\prod_{\nu=1}^{k}\big\{1+B_\nu\big(1+\varphi(\varepsilon,t_\nu-t_{\nu-1})\big)\big\}V(0), \quad t_k<t\leqslant t_{k+1},$$

which can be written down as

$$V(t) \leqslant \prod_{\nu=1}^{i(0,t)} \bigl\{1 + B_\nu\bigl(1 + \varphi(\varepsilon, t_\nu - t_{\nu-1})\bigr)\bigr\} V(0), \quad t > 0.$$

The last inequality implies

$$\begin{aligned}\sum_{i=1}^{m} y_i(t) &\leqslant \prod_{\nu=1}^{i(0,t)} \bigl\{1 + B_\nu\bigl(1 + \varphi(\varepsilon, t_\nu - t_{\nu-1})\bigr)\bigr\} \\ &\quad \times \sum_{i=1}^{m}\left(y_i(0) + \sum_{j=1}^{m} |b_{ij}| L_j \int_0^\infty K_{ij}(s) s e^{\varepsilon s}\, ds \sup_{w\in(-\infty,0]} y_j(w)\right) \\ &= \prod_{\nu=1}^{i(0,t)} \bigl\{1 + B_\nu\bigl(1 + \varphi(\varepsilon, t_\nu - t_{\nu-1})\bigr)\bigr\} \\ &\quad \times \sum_{i=1}^{m}\left(1 + L_i \sum_{j=1}^{m} |b_{ji}| \int_0^\infty K_{ji}(s) s e^{\varepsilon s}\, ds\right) \sup_{w\in(-\infty,0]} y_j(w), \quad t > 0,\end{aligned}$$

from which it follows that

$$\begin{aligned}\sum_{i=1}^{m}\bigl|x_i(t) - x_i^*\bigr| &\leqslant \beta e^{-\varepsilon t} \prod_{\nu=1}^{i(0,t)} \bigl\{1 + B_\nu\bigl(1 + \varphi(\varepsilon, t_\nu - t_{\nu-1})\bigr)\bigr\} \\ &\quad \times \sum_{i=1}^{m} \sup_{w\in(-\infty,0]} \bigl|x_i(w) - x_i^*\bigr| \end{aligned} \tag{2.22}$$

for $t > 0$ and β given by (2.16). □

In general, the estimate (2.22) does not imply any sort of stability of the equilibrium point $\mathbf{x}^*$ of system (2.14), (2.15). In order to prove stability we have to impose some conditions on the moments and amplitudes of the impulse effects.

First we note the obvious

Corollary 2.6. *Let all assumptions of Theorem* 2.5 *hold. If there is a finite number of moments of impulse effect, the equilibrium solution* $\mathbf{x}^*$ *of system* (2.14), (2.15) *is globally asymptotically stable.*

For the case of infinitely many moments of impulse effect we introduce the following additional assumptions:

(H6) $\sup_{k\in\mathbb{Z}^+}(t_k - t_{k-1}) < \infty$.

Suppose that

$$T > t_k - t_{k-1} \quad \forall k \in \mathbb{Z}^+. \tag{2.23}$$

Since $\lim_{\varepsilon\to 0+}\varphi(\varepsilon, z) = z$, we can choose $\varepsilon \in (0, \varepsilon^*)$ so that

$$\varphi(\varepsilon, t_k - t_{k-1}) \leqslant T \quad \forall k \in \mathbb{Z}^+. \tag{2.24}$$

(H7) $B = \sup_{k\in\mathbb{Z}^+} B_k(1 + T) < \infty$.
(H8) $p = \limsup_{t\to\infty}(i(0, t)/t) < \infty$.

Corollary 2.7. *Let all conditions of Theorem* 2.5 *hold. Suppose that the impulse effects* (2.15) *satisfy the assumptions* (H6)–(H8)*, where* T *is defined by* (2.23)*. If* $\varepsilon \in (0, \varepsilon^*)$ *is chosen to satisfy* (2.24) *and*

$$\varepsilon - p\ln(1 + B) > 0, \tag{2.25}$$

then for any number $\tilde\varepsilon$ *satisfying* $0 < \tilde\varepsilon < \varepsilon - p\ln(1 + B)$ *and any compact neighbourhood* K *of the equilibrium* $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_m^*)^T$ *there exists a constant* $\beta \geqslant 1$ *such that all solutions* $\mathbf{x}(t)$ *of* (2.14)*,* (2.15) *with* $\mathbf{x}(w) \in K$*,* $w \in (-\infty, 0]$*, satisfy the inequality*

$$\sum_{i=1}^{m}\bigl|x_i(t) - x_i^*\bigr| \leqslant \beta e^{-\tilde\varepsilon t}\sum_{i=1}^{m}\sup_{w\in(-\infty,0]}\bigl|x_i(w) - x_i^*\bigr| \quad \forall t > 0. \tag{2.26}$$

Proof. First let us note that the conditions of Corollary 2.7 can be satisfied. Indeed, let us suppose that the moments of impulse effect satisfy the conditions (H6) and (H8). Let us fix ε satisfying (2.24). Then we can take the constants B_k, $k \in \mathbb{Z}^+$, so small that B defined by the condition (H7) satisfies the inequality (2.25).

Now suppose that all conditions of Corollary 2.7 hold. By virtue of (H6), (2.23), (2.24) and (H7), inequality (2.22) takes on the form

$$\sum_{i=1}^{m}\bigl|x_i(t) - x_i^*\bigr| \leqslant \beta e^{-\varepsilon t}(1 + B)^{i(0,t)}\sum_{i=1}^{m}\sup_{w\in(-\infty,0]}\bigl|x_i(w) - x_i^*\bigr|.$$

According to condition (H8) for any $\eta > 0$ the inequality $i(0, t) \leqslant (p + \eta)t$ is satisfied for all t large enough. Then we have

$$e^{-\varepsilon t}(1 + B)^{i(0,t)} = \exp\bigl(-\varepsilon t + i(0, t)\ln(1 + B)\bigr) \leqslant \exp\bigl(-\bigl(\varepsilon - (p + \eta)\ln(1 + B)\bigr)t\bigr).$$

Let us take $\tilde\varepsilon < \varepsilon - p\ln(1 + B)$, i.e., $(\varepsilon - \tilde\varepsilon)/\ln(1 + B) > p$. If we denote $\eta = (\varepsilon - \tilde\varepsilon)/\ln(1 + B) - p > 0$, then $\varepsilon - (p + \eta)\ln(1 + B) = \tilde\varepsilon$ and the estimate (2.26) is valid for all t large enough. If we increase the constant β, we can provide the validity of the estimate (2.26) for all $t > 0$. The constant β may still depend on the solution $x(t)$ of system (2.14), (2.15), i.e., on the initial function $\psi(w)$. Because of the compactness of K we can choose β so that (2.26) is valid for all solutions $\mathbf{x}(t)$ such that $\mathbf{x}(w) \in K$ for $w \in (-\infty, 0]$. □

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